PROPAGATION OF DISCONTINUITY WAVES IN COMPLEX RHEOLOGICAL MEDIA WITH VISCOUS PROPERTIES

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The propagation of discontinuity waves of various order in rheological media is examined. It is assumed that the region of discontinuity of values can be represented by an intermediate layer of infinitesimal thickness. By means of this representation, results can be obtained for a rather wide class of continuous media with viscous properties, which generalize Duhem's results. The first integrals of the laws of momentum and energy conservation are obtained, which hold inside the intermediate layer at a shock wave.

It is shown that when viscosity elements are introduced in a special way into the rheological model of a continuous medium, discontinuity waves of any order are propagated in the medium, and that at the surface of a strong discontinuity in a heat-conducting medium, the temperature is continuous. Additional conditions for strain discontinuities at the viscosity elements are obtained. For certain inclusions of the viscosity elements into the rheological model discontinuity waves do not propagate; instead there is merely a weak discontinuity surface which acts as an interface between the flow region of the continuous medium and the region in the state of rest. Contact discontinuities can occur in any continuous medium.

The possible existence of a geometrical discontinuity surface in a viscous gas was examined first by Duhem [1]. He established that singluar strong-discontinuity surfaces cannot take place in a viscous gas. However, if one assumes that the velocity and temperature are continuous in the passage through a singular surface, only contact discontinuities are possible [2].

1. All the following considerations will refer to a rectangular system of Cartesian coordinates x_i. Double subscripts indicate addition, while a subscript behind a comma means partial differentiation with respect to the corresponding coordinate.

Let the rheological model of a continuous medium R consist of m elasticity elements and n viscosity (V_a) elements which are connected in series and in parallel in a certain prescribed order [3]. The stress tensor s_{ij}^a at a V_a element is related to the strain rate tensor ε_{ij}^a at this element by the linear relation

$$s_{ij}^a = \xi_a \varepsilon_{kk}^a \delta_{ij} + 2\eta_a \varepsilon_{ij}^a \qquad (a = 1, 2, \ldots, n), \tag{1.1}$$

where ξ_a, η_a are two viscosity coefficients which, as all the other physical coefficients, are assumed to be constant.

It has been found that the viscous properties of a continuous medium place additional constraints on the propagation capability of discontinuity waves. In order to determine these constraints, a strong-discontinuity surface of rheological quantities is replaced by an infinitely thin intermediate layer of thickness 2h. Inside the thin intermediate layer, a continuous variation of the discontinuous quantities is substituted for a jumpwise variation (Fig. 1).

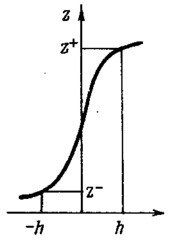

Fig. 1

The existence of an intermediate shock layer is usually attributed [4, 5] to the dissipative properties of the medium, in the sense that they manifest themselves strongly only inside the shock layer and are negligible in the flow region of the medium beyond this layer.

The dissipative properties of the medium can be described by connecting in parallel a certain dissipative rheological model D, on which acts a stress d_{ij}, to the basic rheological model R (Fig. 2).

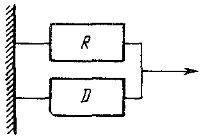

Fig. 2

We introduce a mobile system of rectangular coordinates in such a way that its origin moves at a velocity G together with the discontinuity surface Σ. We direct the x_3-axis along the normal to this surface at an arbitrary mass point on Σ. Then the x_1- and x_2-axes come to lie in the plane tangential to the discontinuity surface. Let the Greek subscripts α and β take the values 1 and 2, and the Latin subscripts i, j, and k the values 1, 2, and 3. All quantities are evaluated in a fixed system of coordinates and are projected onto the axes of the mobile system.

In all the expressions encountered, we separate the derivatives of quantities along the normal to the discontinuity surface from the derivatives along the tangential directions, and substitute a δ-derivative [6] for the partial derivative with respect to time. To this end we write the relations

$$\frac{\partial}{\partial x_i} = \delta_{i3}\frac{\partial}{\partial x_3} + \delta_{i\alpha}\frac{\partial}{\partial x_\alpha}, \quad \frac{\partial}{\partial t} = \frac{\delta}{\delta t} - G\frac{\partial}{\partial x_3}. \tag{1.2}$$

In a mobile system of coordinates, with the aid of (1.2), we obtain the equations of mass and motion conservation for a continuous medium, respectively, in the form

$$\{\rho(v_3 - G)\}_{,3} + \frac{\delta\rho}{\delta t} + (\rho v_\alpha)_{,\alpha} = 0, \tag{1.3}$$

$$\rho(v_3 - G)v_{i,3} + \rho v_\alpha v_{i,\alpha} + \rho\frac{\delta v_i}{\delta t} = \sigma_{i3,3} + \sigma_{i\alpha,\alpha}. \tag{1.4}$$

In a fixed system of coordinates, we write the first and second laws of thermodynamics in the form

$$\rho\frac{dU}{dt} = \varkappa T_{,kk} + \sigma_{ij}v_{i,j} + \varepsilon(T), \tag{1.5}$$

$$\varkappa \geqslant 0, \qquad D_a = s_{ij}^a e_{ij}^a \geqslant 0, \qquad D_b = t_{ij}^b e_{ij}^b \geqslant 0, \tag{1.6}$$
$$a = 1, 2, \ldots, n; \; b = 1, 2, \ldots, p.$$

where ρ is the density, ν_i is the velocity of the medium particles, σ_{ij} are the stresses in the medium, U is the intrinsic energy, T is the absolute temperature, $\varkappa$ is the thermal conductivity, ε is the intensity of external heat sources which depend only on the temperature, D_a is the energy dissipation at viscosity elements, D_b is the energy dissipation at plasticity elements, and p is the number of plasticity elements.

In a mobile system of coordinates, with the aid of (1.2), the first law of thermodynamics (1.5) can be written in the form

$$\rho(v_3 - G)U_{,3} + \rho v_\alpha U_{,\alpha} + \rho\frac{\delta U}{\delta t} = \varkappa(T_{,33} + T_{,\alpha\alpha}) + \sigma_{i3}v_{i,3} + \sigma_{i\alpha}v_{i,\alpha} + \varepsilon(T). \tag{1.7}$$

Prior to performing computations, we make the assumption that the quantities ρ, ν_i, U, T, and d_{ij} are modulo limited in the intermediate layer. This applies also to the strain tensor components at any element of the rheological model and to the first invariant of the stress tensor at the plasticity elements. From the condition that the properties of the medium which are described by an element D vanish beyond the intermediate layer, there follows the equality

$$d_{ij}^+ = d_{ij}^- = 0. \tag{1.8}$$

The plus and minus superscripts imply that the corresponding quantity is evaluated at the leading or trailing shock front of the intermediate layer, respectively.

The boundedness of d_{ij} is proved* if the element D is a viscosity element; the boundedness of all other quantities derives from physical considerations.

*See A. D. Chernyshev, Candidate's thesis, Voronezh State University, 1966.

The stress-strain tensor relation at elasticity elements can be written in the form of a generalized Hooke's law [9]. A prescribed piecewise-smooth convex plasticity condition [10] is assumed for each plasticity element.

These assumptions (boundedness of the quantities, a generalized Hooke's law, and the completeness of the plasticity conditions) lead to the boundedness of the stresses t_{ij} at the elasticity and plasticity elements inside the intermediate layer.

By integrating (1.3) across the intermediate layer over the x_3-coordinate from x_3 to h, we obtain

$$\rho (v_3 - G) - \rho^+ (v_3{}^+ - G) = \int_{x_3}^{h} \left\{ \frac{\delta \rho}{\delta t} + (\rho v_\alpha),_\alpha \right\} dx_3. \tag{1.9}$$

From the properties of the δ-derivative with respect to time and the partial derivatives with respect to the directions in the plane tangential to the discontinuity surface [6], on the basis of the assumption of the boundedness of quantities inside the intermediate layer, it follows that the integrand in (1.9) is bounded inside the intermediate layer. Since the integration step is small, and tends to zero when $h \to 0$, from (1.9), to within small values, we have

$$\rho (v_3 - G) = \rho^+ (v_{,3} - G). \tag{1.10}$$

The approximate relation (1.10), which is valid inside the intermediate layer, becomes an exact relation at the limit for $h \to 0$. For a one-dimensional steady flow region, this equation becomes an exact one even for $h \neq 0$. Equation (1.10) can be also obtained directly from the law of mass conservation [4]. By evaluating the left-hand side of (1.10) for $x_3 = -h$, we obtain the well-known discontinuity relation [6]

$$[\rho (v_3 - G)] = 0. \tag{1.11}$$

where the brackets indicate a jump of a quantity at the discontinuity surface.

In the same manner, by integrating (1.4) across the intermediate layer, on the basis of identical considerations, we obtain, correct to within small values, the equality

$$\sigma_{i3} - \rho^+ (v_3{}^+ - G) v_i = \sigma_{i3}{}^+ - \rho^+ (v_3{}^+ - G) v_i{}^+ + \int_{x_3}^{h} \sigma_{i\alpha,\,\alpha} dx_3. \tag{1.12}$$

It will be shown that the integral in the right-hand side of expression (1.12) is a small quantity.

Taking a section of the rheological model (Fig. 2), the total stress σ_{ij} inside the intermediate layer may be represented in the form of the sum of s_{ij}^a stresses at n_1 viscosity elements, and the sum of t_{ij}^b stresses at m_1 elasticity and plasticity elements which have come to lie in the section chosen, added to d_{ij}:

$$\sigma_{ij} = \sum_{a=1}^{n_1} s_{ij}^a + \sum_{b=1}^{m_1} t_{ij}^b + d_{ij}, \qquad n_1 \leqslant n, \quad m_1 \leqslant m. \tag{1.13}$$

If there exists even a single section of the rheological model which does not intersect a single viscosity element, the inclusion of a system of viscosity elements into the rheological model will be termed the internal method. This, for example, is the case for a Maxwellian body [9]. If an arbitrary section of the rheological model intersects at least one viscosity element, such an inclusion of a system of viscosity elements into the rheological model will be termed the external method. In a Kelvin body, for example, the viscosity element is included by the external method. It may be assumed that when a system of viscosity elements is included by the external method, there will exist an α_a such that

$$\Sigma \alpha_a \varepsilon_{ij}^a = \varepsilon_{ij}. \tag{1.14}$$

The coefficients α_a can take the values 0 and 1. Depending on the type of rheological model, several combinations of α_a values are possible. If a system of viscosity elements is included by the internal method, equality (1.14) is not valid. In addition to the internal and external methods, the viscosity elements can be connected in a kinematically dependent or kinematically independent manner. A connection of a system of viscosity elements will be termed kinematically dependent, if there exists between the strain rates at these elements a relationship of the form

$$\Sigma\beta_a \varepsilon_{ij}^a = 0. \tag{1.15}$$

The coefficients β_a can take the values 0, 1, or −1. A version of the dependent connection of viscosity elements is shown in Fig. 3. For this case, the kinematic relationship (1.15) has the form

$$\varepsilon_{ij}^1 + \varepsilon_{ij}^2 - \varepsilon_{ij}^3 - \varepsilon_{ij}^4 = 0, \quad \beta_1 = \beta_2 = -\beta_3 = -\beta_4 = 1. \tag{1.16}$$

The number of kinematic relations of the (1.15) type is equal to the number of closed circuits composed of viscosity elements, such as shown in Fig. 3. It is these closed circuits that constitute the necessary cause for the existence of relations of the (1.15) type. Relations (1.14) and (1.15) can be treated as rheological relations of a kinematic nature for a given continuous medium. If relation (1.15) is not fulfilled, the connection of the system of viscosity elements is kinematically independent.

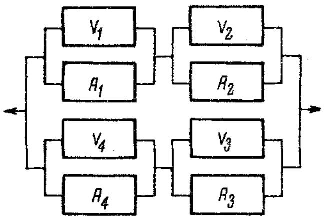

Fig. 3

Let us examine the case of an internal inclusion of a system of n viscosity elements into the rheological model of a continuous medium. For a section that does not intersect any viscosity elements, we obtain from (1.13):

$$\sigma_{ij} = \sum_{b=1}^{m_2} t_{ij}^b + d_{ij}. \tag{1.17}$$

Since the t_{ij}^b and d_{ij} stresses are assumed to be bounded inside the intermediate layer, the total stress σ_{ij} in the continuous medium is also bounded. The integration step in (1.12) is small and, therefore, the integral is a small quantity in this case. Correct to small quantities, equality (1.12) has the form

$$\sigma_{i3} - \sigma_{i3}^+ = \rho^+ (v_3^+ - G)(v_i - v_i^+). \tag{1.18}$$

Evaluating the left-hand side of (1.18) at a given shock front, we obtain the well-known relation in terms of jumps [6]

$$[\sigma_{i3}] = \rho^+ (v_3^+ - G)\ [v_i]. \tag{1.19}$$

Integrating (1.5) across the intermediate layer in the same manner as was done for (1.3) and (1.4), and using for this purpose relations (1.10) and (1.18), with an accuracy to within small quantities, we obtain

$$\rho^+(v_3^+ - G)\{U - U^+ - {}^1/_2 (v_k - v_k^+)^2\} = \varkappa (T_{,3} - T_{,3}^+) + \sigma_{k3}^+(v_k - v_k^+). \tag{1.20}$$

Equations (1.10), (1.18), and (1.20) are the first integrals of the equations of mass and velocity conservation and of the first law of thermodynamics, respectively, which hold for the intermediate layer of a shock wave that propagates in a continuous medium. Equation (1.10) has been obtained earlier, as we have stated; Eq. (1.18) was first obtained in [1], while integral (1.20) is obtained here for the first time. In the special case in which the stress tensor has only a hydrostatic part, these equations were known in [4]. By evaluating the left- and right-hand sides of (1.20), we obtain at the trailing shock front the well-known expression for the law of energy conservation in terms of jumps [8],

$$\rho^+(v_3^+ - G)[U + v_k(v_k^+ - {}^1/_2 v_k)] = \varkappa[T_{,3}] + \sigma_{k3}^+[v_k] \tag{1.21}$$

It is noteworthy that at the limit, for $h \to 0$, Eq. (1.10)–(1.12) as well as (1.18)–(1.21) become exact equations, since the neglected terms tend to zero. Thus, from (1.12), we obtain

$$\lim_{h\to 0} \int_{-h}^{h} \sigma_{i\alpha,\alpha}\, dx_3 = 0 \tag{1.22}$$

Let us choose a section of the rheological model which intersects n_1 viscosity elements, and for which relation (1.13) holds. With the aid of (1.13) and (1.1), equality (1.22) reduces to the form

$$\sum_{a=1}^{n_1} \xi_a b^a_{kk,\,\alpha}\delta_{i\alpha} + 2\eta_a b^a_{i\alpha,\,\alpha} = 0, \quad b^a_{ij} = \lim_{h\to 0} \int_{-h}^{h} \varepsilon^a_{ij}\,dx_3. \tag{1.23}$$

If for a section of the rheological model that does not intersect the viscosity elements, Eq. (1.22) holds unconditionally, for a section that intersects the viscosity elements, relations (1.22) and (1.23) will no longer hold unconditionally. In view of this, relation (1.23) constitutes an additional constraint that is placed on the strain discontinuities at the viscosity elements. Besides 1.23, we obtain from (1.12) one other analogous constraint on the strain discontinuities at the viscosity elements. Let us integrate (1.12) across the intermediate layer from $-h$ to h for section (1.13) and the pass to the limit for $h \to 0$. Taking (1.22) into consideration, at the limit we obtain

$$\sum_{a=1}^{n_1} \xi_a b^a_{kk}\delta_{i3} + 2\eta_a b^a_{i3} = 0. \tag{1.24}$$

There will be just as many relations (1.23) and (1.24) as there are sections of the rheological model which intersect the viscosity elements. It was found that the sum of such sections and relations (1.15) is equal to the number of viscosity elements in the rheological model. To prove this statement, we use the assumption of the boundedness of quantities, together with the rules of constructing rheological models.

Among the six equations (1.23) and (1.24), one is an independent equation. Thus, by differentiating an equation from (1.24) with respect to the coordinate x_α for $i = \alpha$, we arrive at an equation from (1.23) for $i = 3$. A relation can be obtained from the law of energy conservation (1.20). Integrating it with respect to x_3 from $-h$ to h, and passing to the limit for $h \to 0$, we obtain the equality

$$\varkappa\,[T] = 0. \tag{1.25}$$

Thus, in a heat-conducting continuous medium with internal inclusion of viscosity elements, the temperature can not tolerate a discontinuity.

Let us prove that in addition to (1.25), from the laws of thermodynamics one can obtain additional relations of the (1.24) type, which have the form

$$b^a_{kk} = [e^a_{kk}] = 0, \tag{1.26}$$

$$c^a_{ij} = 0, \qquad c^a_{ij} = \lim_{h\to 0} \int_{-h}^{h} dx_3 \int_{x_3}^{h} (\varepsilon^a_{ij})^2 dx_3. \tag{1.27}$$

To prove relations (1.26) and (1 27), the stress intensity of a continuous medium, which figures in the right-hand side of (1.5), is expressed in the form of a sum

$$\sigma_{ij}v_{i,\,j} = \sum_{a=1}^{n} s^a_{ij}\varepsilon^a_{ij} + \sum_{b=1}^{m} t^b_{ij}\varepsilon^b_{ij} + d_{ij}v_{i,\,j}. \tag{1.28}$$

The correctness of this expression can be established by the induction method, with the aid of the rules for constructing rheological models.

We integrate (1.28) across the intermediate layer from x_3 to h, with the aid of (1.18), and we obtain

$$\sigma^+_{k3}[v_k] - \rho^+(v_3{}^+ - G)\left[v_k\left(v_k{}^+ - \frac{1}{2}v_k\right)\right] = \sum_{a=1}^{n}\int_{x_3}^{h} s^a_{ij}\,\varepsilon^a_{ij}\,dx_3 +$$
$$+ \sum_{b=1}^{m}\int_{x_3}^{h} t^b_{ij}\varepsilon_{ij}{}^b\,dx_3 + \int_{x_3}^{h} d_{ij}v_{i,j}\,dx_3. \tag{1.29}$$

We assume that the relationship between the stress tensors eij^{a} and strain rate tensors is described by one of the following expressions:

$$\varepsilon^a_{ij} = \frac{\partial e^a_{ij}}{\partial t}, \qquad \varepsilon^a_{ij} = \frac{de^a_{ij}}{dt},$$
$$\varepsilon^a_{ij} = \frac{De_{ij}}{Dt} = \frac{de_{ij}}{dt} + \frac{1}{2}e^a_{ik}(v_{j,\,k} - v_{k,\,j}) + \frac{1}{2}e^a_{jk}(v_{i,\,k} - v_{k,\,i}). \tag{1.30}$$

Here D/Dt is a covariant time derivative in Jaumann's [11] sense. With the aid of (1.2), we write the expressions in (1.18) in a mobile system of coordinates. The maximum values of $|t_{ij}|$ and $|d_{ij}|$ will be expressed through m_{ij}, which are bounded values. Let us prove the boundedness of the third integral in the right-hand side of expression (1.29):

$$\left|\int_{x_3}^{h} d_{ij} v_{i,j}\, dx_3\right| \leqslant m_{ij} \left|\int_{x_3}^{h} v_{i,j}\, dx_3\right| = m_{i3}\,|v_i^+ - v_i| + \ldots . \tag{1.31}$$

Similarly, we obtain for the second integral in the right-hand side of (1.29) the following expression

$$\left|\int_{x_3}^{h} t_{ij}^b \varepsilon_{ij}^b\, dx_3\right| \leqslant m_{ij} \left|\int_{x_3}^{h} \varepsilon_{ij}^b\, dx_3\right|. \tag{1.32}$$

If we take the first relationship in (1.30), we obtain for the integral in (1.32) the expression

$$\int_{x_3}^{h} \varepsilon_{ij}^b\, dx_3 = G\,(e_{ij}^b - e_{ij}^{b+}) + \ldots . \tag{1.33}$$

For the second relationship in (1.30), with the aid of the mean-value theorem, this integral can be written in the form

$$\int_{x_3}^{h} \varepsilon_{ij}^b\, dx_3 = (v_3{}^* - G)\,(e_{ij}^{b+} - e_{ij}^b) + \ldots . \tag{1.34}$$

With the aid of the third relationship in (1.30), the integral in (1.32) can be reduced to the form

$$\int_{x_3}^{h} \varepsilon_{ij}^b\, dx_3 = (v_3{}^* - G)\,(e_{ij}^{b+} - e_{ij}^b) + {}^1\!/_2\, e_{i3}^{b^*}(v_j{}^+ - v_j) -$$
$$- {}^1\!/_2\, e_{ik}^{b^*}(v_k{}^+ - v_k)\,\delta_{j3} + {}^1\!/_2\, e_{j3}^{b^*}(v_i{}^+ - v_i) - {}^1\!/_2\, e_{jk}^{b^*}(v_k{}^+ - v_k)\,\delta_{i3} + \ldots . \tag{1.35}$$

The ellipsis in expressions (1.31) and (1.33)–(1.35) denotes terms which tend to zero when $h \to 0$, while an asterisk denotes the mean value of a quantity in the intermediate layer. By substituting successively (1.33)–(1.35) into (1.32), we can prove that the last two integrals in (1.29) are modulo limited everywhere within the intermediate layer. Let us integrate (1.29) across the intermediate layer from $-h$ to h. Assuming that all terms in (1.29), with the exception of the first integral in the right-hand side, are bounded, for $h \to 0$ at the limit we obtain

$$\sum_{a=1}^{n} \lim_{h\to 0} \int_{-h}^{h} dx_3 \int_{x_3}^{h} s_{ij}^a \varepsilon_{ij}^a\, dx_3 = 0. \tag{1.36}$$

By virtue of the second law of thermodynamics (1.6), all terms in (1.36) cannot be negative. Therefore, in the case in which equality (1.36) is valid, with the aid of (1.1), we obtain the system of equations

$$(\xi_a + {}^2\!/_3\eta_a) \lim_{h\to 0} \int_{-h}^{h} dx_3 \int_{x_3}^{h} (\varepsilon_{kk}^a)^2\, dx_3 = 0, \tag{1.37}$$

$$\eta_a \lim_{h\to 0} \int_{-h}^{h} dx_3 \int_{x_3}^{h} \left(\varepsilon_{ij}^a - \frac{1}{3}\varepsilon_{kk}^a \delta_{ij}\right)^2 dx_3 = 0. \tag{1.38}$$

Substituting one of the expressions for the strain rate in terms of the strain components from (1.30) into (1.37), we arrive at the relations

$$\lim_{h\to 0} \int_{-h}^{h} dx_3 \int_{x_3}^{h} S\left(\frac{\partial e_{kk}^a}{\partial x_3}\right)^2 dx_3 = 0. \tag{1.39}$$

If the first relationship in (1.30) is used, then $S = G^2$, while for the second and third relations we have $S = (G - v_3)^2$. Here S is a bounded quantity within the intermediate layer in both cases, and hence, in accordance with the mean-value theorem, can be removed from the integral sign. Then, integrating by parts and making use of the mean-value theorem, expression (1.39) can be reduced to the form

$$\lim_{h\to 0} \int_{-h}^{h} dx_3 \int_{x_3}^{h} \left(\frac{\partial e_{kk}^a}{\partial x_3}\right)^2 dx_3 = [e_{kk}^a]\,(\delta^* - {}^1\!/_4\,[e_{nn}^a]) = 0,$$
$$\delta^* = \{{}^1\!/_2\,(e_{kk}^a - e_{kk}^{a+}) - e_{kk}^{a^*}\}^*. \tag{1.40}$$

For $|e_{kk}^a| \neq 0$, the quantity δ^* depends on how the quantity e_{kk}^a changes inside the intermediate layer, which can be adapted to a rapid change in $|e_{kk}^a|$ by varying the properties of the rheological elements. It may be considered, therefore, that

$$\delta^* - {}^1\!/_4[e_{nn}{}^a] \neq 0,$$

in which case, however, there is a discrepancy with equality (1.40). Hence we have a proof of expression (1.26).

By repeating the computations which were used to obtain (1.36) from (1.27), equality (1.38) reduces with the aid of (1.28) to the form (1.29).

Equations (1.26) and (1.27) constitute additional conditions for the propagation of shock waves in continuous media with viscous properties. It is possible to complement this system. It has been stated that the number of equations of the type (1.23) and (1.15) or (1.24) and (1.15) is equal to the number of viscosity elements. This makes it possible by simple transformations with the aid of (1.36) to reduce a system of equations of the type of (1.24) and (1.15) to a closed system of homogeneous linear equations with respect to the quantities b^a_{i3}. By virtue of the independence of this system of equations by definition, we obtain

$$b^a_{i3} = 0. \tag{1.41}$$

If the coupling of the viscosity elements in the rheological model is kinematically independent, the proof of the validity of (1.41) is obvious.

For a kinematically dependent coupling of the viscosity elements, the scheme of the proof of equality (1.41) will be demonstrated for the case shown in Fig. 3.

For simplicity, it is assumed that the elements A_b which experience the stresses t^b_{ij} consist of elasticity and plasticity elements. For such a model, we have the equalities

$$s^1_{ij} + t^1_{ij} = s^2_{ij} + t^2_{ij}, \qquad s^3_{ij} + t^3_{ij} = s^4_{ij} + t^4_{ij}. \tag{1.42}$$

By integrating (1.42) across the intermediate layer for $f = 3$, we obtain

$$\begin{gathered}(\xi_1 b^1_{kk} - \xi_2 b^2_{kk})\,\delta_{i3} + 2\eta_1 b^1_{i3} - 2\eta_2 b^2_{i3} = 0,\\ (\xi_3 b^3_{kk} - \xi_4 b^4_{kk})\,\delta_{i3} + 2\eta_3 b^3_{i3} - 2\eta_4 b^4_{i3} = 0.\end{gathered} \tag{1.43}$$

With the aid of (1.26), equalities (1.43) can be simplified to the form

$$\eta_1 b^1_{i3} - \eta_2 b^2_{i3} = 0, \qquad \eta_3 b^3_{i3} - \eta_4 b^4_{i3} = 0. \tag{1.44}$$

We add to it an equation from (1.24) for a section of the model which intersects the first and third viscosity elements

$$\eta_1 b^1_{i3} + \eta_3 b^3_{i3} = 0. \tag{1.45}$$

We obtain the closing equation in system (1.44), (1.45) by integrating (1.16) across the intermediate layer for $f = 3$:

$$b^1_{i3} + b^2_{i3} - b^3_{i3} - b^4_{i3} = 0. \tag{1.46}$$

The determinant of system (1.44)–(1.46) with respect to b^a_{i3} differs from zero, and therefore $b^a_{i3} = 0$; Q. E. D.

From (1.23) and (1.15), it can be shown that the equality

$$[b^a_{\alpha\beta,\beta}\ ] = 0 \tag{1.47}$$

is fulfilled.

The expressions (1.41) and (1.47), just as (1.27), place constraints on the discontinuities of the strain tensor components at the viscosity elements and their derivatives in directions tangential to the discontinuity surface. If the first or second expression for the strain rate tensor in terms of the strain tensor is used in (1.30), then (1.27), (1.41), and (1.47) can be appreciably simplified. In this case, in exactly the same manner in which (1.26) was obtained from (1.37) and (1.27), we get

$$[e_{ij}^{a}] = 0. \tag{1.48}$$

From (1.41) and (1.47), it follows that solution (1.48) also satisfies these equations. Thus, in the case under consideration, the strain components at all the viscosity elements are continuous at the shock wave surface.

If the third expression for the strain rate tensor in terms of the strain tensor is used in (1.30), then integrals (1.27), (1.41), and (1.47) cannot be evaluated, owing to the presence of nonlinear terms which account for the effect of the rotation of the environment of the medium particle under study.

Generally speaking, equality (1.48) is now no longer valid, so that the strain tensors at the viscosity elements are now discontinuous. These discontinuities are present owing to the nonlinear terms in (1.30), and therefore the fact that the jumps of $|e_{ij}^{a}|$ differ from zero at the shock wave surface should be treated as a secondary effect.

In the extension of an n-th order weak discontinuity surface, the density, velocity, and the strain and stress components at any element of the rheological model, together with their derivatives down to the (n − 1)-th order must be continuous. In order that the strain-component derivatives down to the (n − 1)-th order be continuous at an element V_a, by virtue of (1.1), the (n − 1)-th-order derivatives of the strain rate tensor components at the element V_a or the n-th-order derivatives of the strain tensor components at this element must also be continuous. If a certain system of rheological elasticity and plasticity elements is connected in parallel with a viscosity element in the rheological model, then the strain components and their derivatives to the n-th order will be continuous at this system of elements. Only their higher-order derivatives may be discontinuous. In this case the discontinuity surface is termed neutral [12] with respect to a system of rheological elements connected in parallel with a viscosity element.

The propagation of weak discontinuities must obey dynamic, geometrical, and kinematic conditions for the compatibility of the discontinuities of rheological quantities [6]. As distinct from weak discontinuities at shock waves, conditions (1.27), (1.28), and (1.47) must be fulfilled in addition to the aforesaid conditions. For each expression for the strain rate tensor in terms of the strain tensor in (1.30), the spherical portion of the strain tensor at any viscosity element in the rheological model of a continuous medium is continuous at the shock wave. If the strain rate is defined by a partial or mass time-derivative of the strains, then from the additional conditions, it follows that at viscosity elements, the strain components are continuous at the shock wave. If, on the other hand, the strain rate is defined by a covariant time-derivative of the strains, then the strains at the viscosity elements may be discontinuous. This may be treated as a secondary effect.

2. Let one viscosity element be included in the rheological model of a continuous medium by the external method, while the remaining (n − 1) elements constitute a system connected by the internal method. In this case, there will exist a section of the rheological model that intersects only the first of all the viscosity elements. According to (1.13) and (1.14), the equality

$$\sigma_{ij} = \xi_1 \varepsilon_{kk}\delta_{ij} + 2\eta_1\varepsilon_{ij} + \sum_{b=1}^{m} t_{ij}^{b} + d_{ij}, \quad \varepsilon_{ij}^{1} = \varepsilon_{ij} = {}^1/_2\,(v_{i,j} + v_{j,i}), \tag{2.1}$$

holds for this section.

Now, it is no longer possible to obtain relation (1.23), since any section of the rheological model will intersect viscosity elements, while relations (1.12) and (1.24), where $n_1 = 1$, remain valid, since they were derived from an arbitrary section. Substituting ε_{ij}^{1} into (1.24), we arrive at the equation

$$\xi_1\,[v_3]\,\delta_{i3} + \eta_1\,[v_i + v_3\delta_{i3}] = 0, \tag{2.2}$$

or

$$(\xi_1 + 2\eta_1)\,[v_3] = 0,\ \eta_1\,[v_\alpha] = 0. \tag{2.3}$$

In the general case for $\eta_1 > 0$, from (2.3) we find that the velocity components are continuous at a strong discontinuity surface. From the continuity of the velocity follows the continuity of the distortion vector for $v_3^{+} \neq G$ and, hence, the continuity of the strain tensor. By virtue of (2.3), Eq. (1.19) can be simplified:

$$[\sigma_{i3}] = 0. \tag{2.4}$$

Let us assume that if the strain tensor of a continuous medium is continuous, the strain tensors at any rheological element are also continuous. The same applies to the continuity of the partial derivatives of the strain tensors. From (2.4), it follows that no dynamic forces are generated at the discontinuity surface. If, in this case, the stress tensor components at the plasticity elements are discontinuous, such surfaces are either sliding surfaces [3, 6] or contact discontinuity surfaces. These surfaces do not expand over the medium. By excluding such surfaces from the analysis, we automatically postulate the continuity of stresses at the plasticity elements; i. e., the stress components at the elasticity and plasticity elements, in this case, are continuous at the discontinuity surface under consideration. If the assumed expanding surface is a shock wave, there exists an intermediate layer in which the medium has the same properties as in the neighborhood of shock fronts. Any quantity with a plus or minus exponent can be evaluated as the limiting value of this quantity inside the intermediate layer in the neighborhood of the corresponding shock front. Therefore, with the aid of (1.13), regardless of the state of the medium in front and behind the shock wave, one can calculate from (2.4) the discontinuity

$$\xi_1 [v_{3,3}] \delta_{i3} + \eta_1 [v_{i,3} + v_{3,3}\delta_{i3}] = 0, \tag{2.5}$$

or

$$(\xi_1 + 2\eta_1) [v_{3,3}] = 0, \ \eta_1 [v_{\alpha,3}] = 0. \tag{2.6}$$

With the aid of geometrical conditions for the compatibility of discontinuities, for $\eta_1 > 0$, from (2.6) we find that the components of the tensor $v_{i,j}$ are continuous at the discontinuity surface under consideration. This means that the density and stress components at all the elements are continuous at the discontinuity surface. Consequently, Σ can be solely a weak discontinuity surface. From (2.3) or (2.6), it follows that an equivoluminal strong or weak discontinuity wave, on which $[v_3] = 0$ or $[v_{3,3}] = 0$, respectively, is only possible in the special case for $\eta_1 = 0$. If $(\xi_1 + 2\eta_1) = 0$ then, by virtue of the second law of thermodynamics, it is necessary that $\xi_1 = \eta_1 = 0$, which is equivalent to the absence of the first viscosity element. In this case, we arrive at the case of an internal inclusion of a system of viscosity elements in the rheological model, which was examined in the previous paragraph.

Until now the results obtained in this paragraph were independent of whether the continuous medium is in a state of flow at both sides of the discontinuity surface or only at one of its sides. In the following analysis, however, this question is of great importance.

Let the medium be in a state of flow at both sides of the discontinuity surface and, hence, retain its characteristics in the passage through this surface. In this case, Eq. (1.13), where $n_1 = 1$, continues to hold at both sides of a given discontinuity surface. By substituting (1.13) into the equation of motion (1.4) and evaluating the discontinuity procedure from this equation with allowance for (2.3) and (2.6), we arrive at the equation

$$\xi_1 [v_{3,33}] \delta_{i3} + \eta_1 [v_{i,33} + v_{3,33} \delta_{i3}] = 0. \tag{2.7}$$

With the aid of the geometrical conditions for the compatibility of discontinuities, from (2.7) we find that for $\eta_1 > 0$, the velocity and all its first and second derivatives are continuous at the assumed discontinuity surface. By definition, it follows from here that the strain tensors at all the rheological elements are continuous and also the stress tensors and their first derivatives are continuous at the elasticity and plasticity elements. By successively differentiating (1.44) several times, each time evaluating the discontinuity procedure with allowance for (1.13), (2.3), and (2.6) and for analogous new relations, we find that the density and the velocity, including all its derivatives, are continuous at the assumed discontinuity surface.

Then, by definition, the strain and stress tensors, together with all their derivatives, are continuous at all the rheological elements, i. e., there can be no extension of a discontinuity surface of any order in the case under consideration.

If the medium is in a state of flow only at one side of the discontinuity surface, the properties of the medium manifest themselves only at that side and in the intermediate layer. At the other side of the discontinuity surface, the medium exhibits other properties, according to other governing equations. From here, it follows that there is little sense in evaluating the discontinuity procedure from Eq. (1.13) and from the derivatives of this equation. In view of this, only conditions (2.2) and (2.5), obtained as a result of the introduction of an intermediate layer, continue to be valid. Conditions (2.7) and all the subsequent inferences hold no longer. Thus, in this case, the propagation of weak discontinuities, which form an interface between the flow region of a viscous continuous medium and the stable region [13] can occur. On these surfaces, the velocity, density, deformation of the medium, and all the first derivatives of these quantities are continuous. For $\eta_1 = 0$, an equivoluminal discontinuity surface of arbitrary order can also take place.

It is obvious that (1.25) remains in force, i.e., the temperature cannot be discontinuous in a heat-conducting continuous medium.

If $G = v_3^+$, even under the condition that strain and stress tensor components at all the rheological elements are continuous, we find from (1.8) that a density discontinuity is possible, i. e., that contact discontinuities can occur for any type of inclusion of viscosity elements into the rheological model of a continuous medium.

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